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Eigenvalue correlations in non-Hermitean symplectic random matrices

Eugene Kanzieper

Hitachi Cambridge Laboratory, Madingley Road, Cambridge CB3 0HE, UK
and
Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot 76100, Israel

E-mail: eugene@phy.cam.ac.uk and eugene.kanzieper@weizmann.ac.il

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Abstract
A correlation function of complex eigenvalues of $N \times N$ random matrices drawn from non-Hermitean random matrix ensembles of symplectic symmetry is given in terms of a quaternion determinant. Spectral properties of Gaussian ensembles are studied in detail in the regimes of weak and strong non-Hermiticity.

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1. Introduction

Statistical ensembles of generic real, complex and quaternion matrices were first introduced in the pioneering work by Ginibre (1965) who managed to derive the joint probability distribution function (jpdf) of N complex eigenvalues $\{z_\ell\} = \{x_\ell + \mathrm{i}y_\ell\}$ of $N \times N$ complex ($\beta = 2$) and quaternion ($\beta = 4$) non-Hermitean random matrices:

$$P_N^{(2)}(z_1, \ldots, z_N) = C_2(N) \prod_{k<\ell} |z_k - z_\ell|^2 \prod_{\ell=1}^{N} w^2(z_\ell, \bar{z}_\ell) \tag{1}$$

$$P_N^{(4)}(z_1, \ldots, z_N) = C_4(N) \prod_{k<\ell} |z_k - z_\ell|^2 |z_k - \bar{z}_\ell|^2 \prod_{\ell=1}^{N} |z_\ell - \bar{z}_\ell|^2 w^2(z_\ell, \bar{z}_\ell) \tag{2}$$

where $C_\beta(N)$ is a normalization constant, $w^2(z, \bar{z})$ is a weight function (see discussion below). For real matrices ($\beta = 1$) with no further symmetries, the reader is referred to much later papers by Lehmann and Sommers (1991), and also by Edelman (1997).

Although Ginibre's derivation of equations (1) and (2) holds for random matrices with Gaussian distributed entries, that is for

$$w^2(z, \bar{z}) = w_0^2(z, \bar{z}) = \mathrm{e}^{-z\bar{z}} \tag{3}$$

we will allow the weight $w^2(z,\bar z)$ to be an arbitrary benign function of z and $\bar z$ provided the normalization $C_\beta(N)$ exists. It should be emphasized[1] that such an innocent (at first glance) extension is quite nontrivial as it raises the question about the existence of an underlying matrix model whose eigenvalue representation would coincide with equations (1) and (2). Suppose that the underlying non-Hermitean matrix commutes with its adjoint (such matrices are often called normal matrices, see, e.g., Oas 1997), the interpretation of equations (1) and (2) as a non-Gaussian jpdf is correct. Note, however, that such a commutativity constraint is not a must. For example, an ensemble of weakly non-Hermitean matrices introduced by Fyodorov *et al* (1997) is described by jpdf of forms (1) and (2), see Fyodorov *et al* (1998) and Hastings (2000).

Of particular interest is the n-point correlation function which describes a probability density to find n complex eigenvalues around each of the points $z_1,\ldots,z_n$ while positions of the remaining levels are unobserved:

$$R_n^{(\beta)}(z_1,\ldots,z_n)=\frac{N!}{(N-n)!}\int\cdots\int \mathrm{d}^2Z_{n+1}\ldots\mathrm{d}^2Z_N\,P_N^{(\beta)}(z_1,\ldots,z_N). \tag{4}$$

The integration measure d^2Z_ℓ is $\mathrm{d}^2Z_\ell=\mathrm{d}x_\ell\,\mathrm{d}y_\ell$. Quite often, one is also interested in a thermodynamic limit

$$\rho_n^{(\beta)}(z_1,\ldots,z_n)=\lim_{N\to\infty}\frac{1}{\delta_N^{2n}}R_n^{(\beta)}\left(\frac{z_1}{\delta_N},\ldots,\frac{z_n}{\delta_N}\right) \tag{5}$$

which magnifies spectrum resolution on the appropriate energy scale δ_N while letting matrix size N tend to infinity.

At $\beta=2$, the n-point correlation function for non-Hermitean matrix model has also been studied by Ginibre (1965). Adopting the method of orthogonal polynomials introduced in the context of Hermitean random matrix theory by Mehta and Gaudin (1960), it is a straightforward exercise to demonstrate that $R_n^{(2)}(z_1,\ldots,z_n)$ admits the determinant representation

$$R_n^{(2)}(z_1,\ldots,z_n)=\det\left[K_N^{(2)}(z_k,\bar z_\ell)\right]_{k,\ell=1,\ldots,n}.$$

The scalar kernel

$$K_N^{(2)}(z,z')=w(z,\bar z)w(z',\bar z')\sum_{k=0}^{N-1}P_k(z)P_k(z')$$

is expressed in terms of polynomials $P_k(z)$ orthonormal in the complex plane $z=x+\mathrm{i}y$

$$\int \mathrm{d}^2Z\,w^2(z,\bar z)P_k(z)P_\ell(\bar z)=\delta_{k\ell}$$

with respect to the measure $w^2(z,\bar z)\,\mathrm{d}^2Z$.

For instance, the density of states and the two-point correlation function equal

$$R_1^{(2)}(z)=K_N(z,\bar z)$$

and

$$R_2^{(2)}(z_1,z_2)=K_N(z_1,\bar z_1)K_N(z_2,\bar z_2)-|K_N(z_1,\bar z_2)|^2$$

respectively.

Non-Hermitean random matrices at $\beta=4$ have also received some attention in both physical and mathematical literature especially following a recent burst of interest in spectral properties of non-Hermitean random operators (see, e.g., Efetov 1997).

[1] The referee is thanked for pointing this out.

Mehta (1967) considered a non-Hermitean matrix model of symplectic symmetry for Ginibre's weight function $w_0^2(z,\bar{z}) = \exp(-z\bar{z})$, and established a quaternion determinant structure of one- and two-point correlation functions. He also conjectured a similar structure to hold for all n-point correlation functions. In Ginibre's case, these appear in the revised 1991 edition of Mehta's book.

Further progress has come with the development of field theoretic techniques. Kolesnikov and Efetov (1997), driven by possible applications in quantum chromodynamics (Halasz *et al* 1997), have formulated a nonlinear supersymmetry σ-model for this class of random matrices, and derived an expression for the eigenvalue density in a somewhat richer model (see equation (21) below).

More recently, yet another field theory approach (also known as the replica method) was outlined by Nishigaki and Kamenev (2002). There, the well-known Mehta expressions were reproduced for one-point correlation function in the case of the very same Ginibre's weight function $w_0^2(z,\bar{z})$. Unfortunately, both the mentioned techniques run into obstacles when one attempts to study higher-order correlation functions whilst replica σ-models seem to reliably provide asymptotic expansions only.

A different route has been chosen by Hastings (2000) who suggested that there exists a mapping of non-Hermitean random matrices of symplectic symmetry onto a fermion field theory. Even though the method might have been potentially applicable to a study of n-point correlation functions in the *bulk* of a complex spectrum, these have not explicitly been worked out beyond the two-point correlation function.

Our paper reports on a comprehensive treatment of integrable structure of non-Hermitean random matrix models at $\beta = 4$. It sets a transparent and coherent framework to study all n-point correlation functions: while easily applied to reproduce the results of the aforementioned studies and extend them to higher-order correlation functions in the spectrum bulk, it may go much farther and serve as a proper starting point to explore eigenvalue correlations near the spectrum edges and/or address the issue of universality (for a recent review of the universality phenomenon in the context of Hermitean random matrix models see, e.g., Kanzieper and Freilikher 1999).

The paper is organised as follows. Section 2 announces a most general form of n-point correlation function whatever the weight function in equation (2) is. A proof is given in section 3. A concept of skew orthogonal polynomials which are central to performing explicit calculations is elaborated in detail in section 4. There, exact expressions for skew orthogonal polynomials are given in terms of multi-fold integrals. For the Gaussian weight (equation (21)), the polynomials are evaluated explicitly. In section 5, n-point correlation functions for the Gaussian weight (equation (21)) are derived for finite N as well as in the large-N limit. Section 6 contains concluding remarks and briefly mentions further possible applications of the formalism developed.

2. Correlation function at β = 4 and eigenvalue depletion along the real axis

For a symplectic ensemble, the following representation holds for the n-point correlation function:

$$R_n^{(4)}(z_1,\ldots,z_n) = Q\det\left[K_N^{(4)}(z_k,z_\ell)\right]_{k,\ell=1,\ldots,n}. \tag{6}$$

Here $Q\det$ stands for a quaternion determinant (Dyson 1972). The 2×2 matrix kernel

$$K_N^{(4)}(z,z') = (\bar{z}-z)^{1/2}(\bar{z}'-z')^{1/2}w(z,\bar{z})w(z',\bar{z}')\begin{pmatrix}\kappa_N(\bar{z},z') & -\kappa_N(\bar{z},\bar{z}')\\ \kappa_N(z,z') & -\kappa_N(z,\bar{z}')\end{pmatrix} \tag{7}$$

where the 'prekernel' κ_N is

$$\kappa_N(z, z') = \sum_{k,\ell=0}^{2N-1} p_k(z)(M^{-1})_{k\ell}\, p_\ell(z') \tag{8}$$

and M^{-1} is an inverse to the real antisymmetric matrix M with the entries

$$M_{k\ell} = \int \mathrm{d}^2 Z(\bar{z} - z) w^2(z, \bar{z})\, [p_k(z) p_\ell(\bar{z}) - p_\ell(z) p_k(\bar{z})]. \tag{9}$$

The polynomials $p_k(z)$ are arbitrary provided the inverse M^{-1} exists. Since the matrix M is antisymmetric, the formulae would be simplest had it contained N copies of the 2×2 matrix

$$\mathrm{i}\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

along the main diagonal. This is achieved by letting $p_k(z)$ be skew-orthogonal polynomials $q_k(z)$ in the complex domain:

$$\langle q_{2k+1}, q_{2\ell} \rangle_S = -\langle q_{2\ell}, q_{2k+1} \rangle_S = r_k \delta_{k\ell} \tag{10}$$

$$\langle q_{2k+1}, q_{2\ell+1} \rangle_S = \langle q_{2k}, q_{2\ell} \rangle_S = 0. \tag{11}$$

The skew product $\langle f, g \rangle_S$ is defined as[2]

$$\langle f, g \rangle_S = \int \mathrm{d}^2 Z(\bar{z} - z) w^2(z, \bar{z})\, [f(z) g(\bar{z}) - f(\bar{z}) g(z)]. \tag{12}$$

With this choice in mind, the prekernel κ_N further simplifies to

$$\kappa_N(z, z') = \sum_{k=0}^{N-1} \frac{q_{2k+1}(z) q_{2k}(z') - q_{2k+1}(z') q_{2k}(z)}{r_k}. \tag{13}$$

In particular, the density of states and the two-point correlation functions are expressed as

$$R_1^{(4)}(z) = (\bar{z} - z) w^2(z, \bar{z}) \kappa_N(z, \bar{z}) \tag{14}$$

and

$$\begin{aligned} R_2^{(4)}(z_1, z_2) = {} & (\bar{z}_1 - z_1)(\bar{z}_2 - z_2) w^2(z_1, \bar{z}_1) w^2(z_2, \bar{z}_2) \\ & \times \left[\kappa_N(z_1, \bar{z}_1)\kappa_N(z_2, \bar{z}_2) - |\kappa_N(z_1, z_2)|^2 + |\kappa_N(z_1, \bar{z}_2)|^2\right] \end{aligned} \tag{15}$$

respectively.

Note that, in accordance with our solution (equations (6) and (7)), the n-point correlation function universally vanishes along the real axes $\mathrm{Im}\, z_\ell = 0$,

$$R_n^{(4)}(z_1, \ldots, z_n) \propto \prod_{\ell=1}^{n} [\mathrm{Im}\, z_\ell]^\alpha \qquad \alpha \equiv 2$$

whatever the weight function $w^2(z, \bar{z})$ is. It is this specific feature of spectral correlations in symplectic ensembles of non-Hermitean random matrices that has been revealed, for $n = 1$, in early numerical simulations due to Halasz *et al* (1997). Qualitatively, such a depletion of complex eigenvalues along the real axis might have been anticipated after a brief inspection of both the jpdf (equation (2)) and the definition of the n-point correlation function (equation (4)).

The results announced (equations (6)–(13)) will be derived in section 3. In section 4, we study properties of skew-orthogonal polynomials which constitute a natural basis to perform actual calculations of spectral fluctuations in $\beta = 4$ non-Hermitean random matrix ensembles. The latter are addressed in section 5, where we consider an ensemble of $N \times N$ random matrices associated with a Gaussian measure. Correlation functions in the regimes of strong (Ginibre 1965) and weak (Fyodorov *et al* 1997) non-Hermiticities are explicitly given there for finite N as well as in the limit of infinite matrices.

[2] Note a difference from the skew orthogonality arising in the context of Hermitean random matrices (Mahoux and Mehta 1991).

3. Derivation

To derive a quaternion determinant representation of n-point correlation function, we will follow an elegant idea of Tracy and Widom (1998). These authors have introduced generating functional

$$G[f] = \int \cdots \int \mathrm{d}^2 Z_1, \ldots, \mathrm{d}^2 Z_N P_N(z_1, \ldots, z_N) \prod_{k=1}^{N} [1 + f(z_k)]$$

such that the n-point correlation function $R_n(z_1, \ldots, z_n)$ defined by equation (4) can be viewed as the coefficient of $\alpha_1 \ldots \alpha_n$ in the expansion of $G[f]$ for a particular choice $f(z) = \sum_{r=1}^{N} \alpha_r \delta^2(z - z_r)$. Assuming that $G[f]$ admits the representation

$$G[f] = \sqrt{\det(I + K_N f)}$$

where K_N denotes the operator with 2×2 (self-dual) matrix kernel $K_N(z, z')$ and f denotes multiplication by that function, Tracy and Widom (1998) have explicitly evaluated the coefficient of $\alpha_1, \ldots, \alpha_n$ and found it to be equal to the quaternion determinant in the rhs of equation (6).

Hence, in accordance with this statement (which we will name the Tracy–Widom theorem), one has to seek a suitable representation for $G[f]$ with the jpdf given by equation (2). This is easy. Due to the identity

$$\prod_{k<\ell} (x_k - x_\ell) \prod_{k,\ell=1}^{N} (y_k - x_\ell) \prod_{k>\ell} (y_k - y_\ell) = \det \begin{bmatrix} x_\ell^{k-1} \\ y_\ell^{k-1} \end{bmatrix}_{\substack{k=1,\ldots,2N \\ \ell=1,\ldots,N}}$$

one notes that

$$\prod_{k<\ell} |z_k - z_\ell|^2 |z_k - \bar{z}_\ell|^2 \prod_{\ell=1}^{N} (\bar{z}_\ell - z_\ell) = \det \begin{bmatrix} z_\ell^{k-1} \\ \bar{z}_\ell^{k-1} \end{bmatrix}_{\substack{k=1,\ldots,2N \\ \ell=1,\ldots,N}}.$$

To derive the latter, we have put $x_\ell = z_\ell$ and $y_\ell = \bar{z}_\ell$ in the former. With this result in mind, the jpdf $P_N^{(4)}$ can be cast into the form

$$P_N^{(4)}(z_1, \ldots, z_N) = C_4(N) \prod_{\ell=1}^{N} (z_\ell - \bar{z}_\ell) w^2(z_\ell, \bar{z}_\ell) \det \begin{bmatrix} p_{k-1}(z_\ell) \\ p_{k-1}(\bar{z}_\ell) \end{bmatrix}_{\substack{k=1,\ldots,2N \\ \ell=1,\ldots,N}}.$$

Here, we have replaced the sequence of monomials $\{z_\ell^k\}$ by arbitrary monic polynomials $\{p_k(z_\ell)\}$ of degree k as this leaves the value of determinant intact. If the polynomials p_k were not monic, the normalization prefactor $C_4(N)$ would change.

This representation is fairly useful due to de Bruijn's (1955) integration formula

$$\int \cdots \int \mathrm{d}\alpha(Z_1) \cdots \mathrm{d}\alpha(Z_N) \det \begin{bmatrix} f_k(z_\ell) \\ g_k(z_\ell) \end{bmatrix}_{\substack{k=1,\ldots,2N \\ \ell=1,\ldots,N}}$$

$$= (2N)! \mathrm{Pf} \left[\int \mathrm{d}\alpha(Z) [f_k(z) g_\ell(z) - f_\ell(z) g_k(z)] \right]_{k,\ell=1,\ldots,2N}$$

in which $d\alpha(Z)$ is an integration measure and 'Pf' stands for Pfaffian. One derives the following chain:

$$G^2[f] \propto \det\left[\int d^2Z(\bar{z}-z)w^2(z,\bar{z})[1+f(z)][p_k(z)p_\ell(\bar{z})-p_\ell(z)p_k(\bar{z})]\right]$$

$$\propto \det\left[M_{k\ell}+\int d^2Z(\bar{z}-z)w^2(z,\bar{z})f(z)[p_k(z)p_\ell(\bar{z})-p_\ell(z)p_k(\bar{z})]\right] \quad (16)$$

$$\propto \det\left[\delta_{k\ell}+\int d^2Z(\bar{z}-z)w^2(z,\bar{z})f(z)[\pi_k(z)p_\ell(\bar{z})-p_\ell(z)\pi_k(\bar{z})]\right]. \quad (17)$$

Matrix M in equation (16) is defined by equation (9). To derive equation (17), we have factored out M on the left. Note that equation (17) automatically bears a proper normalization $G^2[0] \equiv 1$. The polynomials π_k are

$$\pi_k(z) = \sum_{\ell=0}^{2N-1}(M^{-1})_{k\ell}\,p_\ell(z).$$

The matrix appearing under the sign of determinant in equation (17) can be represented as $I + AB$ with

$$A(\ell,z) = f(z)(\bar{z}-z)^{1/2}w(z,\bar{z})(\pi_\ell(z), -\pi_\ell(\bar{z}))$$

$$B(z,\ell) = (\bar{z}-z)^{1/2}w(z,\bar{z})\begin{pmatrix} p_\ell(\bar{z}) \\ p_\ell(z)\end{pmatrix}.$$

As soon as the transposition does not affect the value of the determinant, one observes the identity $\det(I+AB) = \det(I+BA)$. In this case BA is the integral operator with matrix kernel $K_N(z_1,z_2)f(z_2)$ where

$$K_N(z_1,z_2) = (\bar{z}_1-z_1)^{1/2}(\bar{z}_2-z_2)^{1/2}w(z_1,\bar{z}_1)w(z_2,\bar{z}_2) \times \begin{pmatrix} \sum_\ell p_\ell(\bar{z}_1)\pi_\ell(z_2) & -\sum_\ell p_\ell(\bar{z}_1)\pi_\ell(\bar{z}_2) \\ \sum_\ell p_\ell(z_1)\pi_\ell(z_2) & -\sum_\ell p_\ell(z_1)\pi_\ell(\bar{z}_2)\end{pmatrix}.$$

Hence, we have proved that $G^2[f] = \det(I + K_N f)$. Since the 2×2 matrix K_N is self-dual[3], equation (6) follows by virtue of the Tracy–Widom theorem. This completes our proof.

4. Skew-orthogonal polynomials

4.1. General weight $w^2(z,\bar{z})$

We have seen in section 2 that skew-orthogonal polynomials defined by equations (10)–(12) represent a natural basis in which calculations become simplest. These can explicitly be found

[3] Indeed, since the quaternion κ_N is represented by 2×2 matrix

$$\theta[\kappa_N] = \begin{pmatrix} \kappa_N(\bar{z}_1,z_2) & -\kappa_N(\bar{z}_1,\bar{z}_2) \\ \kappa_N(z_1,z_2) & -\kappa_N(z_1,\bar{z}_2)\end{pmatrix}$$

the dual quaternion $\tilde{\kappa}_N$ is given by

$$\theta[\tilde{\kappa}_N] = \begin{pmatrix} -\kappa_N(z_1,\bar{z}_2) & \kappa_N(\bar{z}_1,\bar{z}_2) \\ -\kappa_N(z_1,z_2) & \kappa_N(\bar{z}_1,z_2)\end{pmatrix}.$$

Self-duality is a consequence of the equality $\sigma_y\theta[\tilde{\kappa}_N^T] = \theta[\kappa_N]\sigma_y$.

for a general weight function $w^2(z,\bar{z})$ provided the integrals below make sense and, in a monic normalization, are given by the following $2n$-fold integrals:

$$q_{2n}(z) \equiv \frac{1}{A_n}\int\cdots\int \mathrm{d}^2Z_1\cdots\mathrm{d}^2Z_n \prod_{\ell=1}^{n}(z-z_\ell)(z-\bar{z}_\ell) \times \prod_{k<\ell}|z_k-z_\ell|^2|z_k-\bar{z}_\ell|^2 \prod_{\ell=1}^{n}|z_\ell-\bar{z}_\ell|^2 w^2(z_\ell,\bar{z}_\ell), \tag{18}$$

$$q_{2n+1}(z) \equiv \frac{1}{A_n}\int\cdots\int \mathrm{d}^2Z_1\cdots\mathrm{d}^2Z_n \prod_{\ell=1}^{n}(z-z_\ell)(z-\bar{z}_\ell)\left(z+\sum_{k=1}^{n}(z_k+\bar{z}_k)+c_n\right) \times \prod_{k<\ell}|z_k-z_\ell|^2|z_k-\bar{z}_\ell|^2 \prod_{\ell=1}^{n}|z_\ell-\bar{z}_\ell|^2 w^2(z_\ell,\bar{z}_\ell). \tag{19}$$

Here c_n is an arbitrary constant which we will set to zero, $c_n = 0$, whilst

$$A_n = \int\cdots\int \mathrm{d}^2Z_1\cdots\mathrm{d}^2Z_n \prod_{k<\ell}|z_k-z_\ell|^2|z_k-\bar{z}_\ell|^2 \prod_{\ell=1}^{n}|z_\ell-\bar{z}_\ell|^2 w^2(z_\ell,\bar{z}_\ell). \tag{20}$$

For similar representations of skew-orthogonal polynomials arisen in the context of Hermitean random matrix theory see Eynard (2001).

To prove that equations (18) and (19) obey skew-orthogonality relations (equations (10)–(12)), it is sufficient to show that (i) $\langle q_{2n}, z^m\rangle_S = 0$ and (ii) $\langle q_{2n+1}, z^m\rangle_S = 0$ for integer $0 \leqslant m \leqslant 2n-1$.

(i) Consider

$$\langle q_{2n}, z^m\rangle_S \propto \int \mathrm{d}^2Z(\bar{z}-z)w^2(z,\bar{z})\int\cdots\int \mathrm{d}^2Z_1\cdots\mathrm{d}^2Z_n \times \prod_{k<\ell}|z_k-z_\ell|^2|z_k-\bar{z}_\ell|^2 \prod_{\ell=1}^{n}|z_\ell-\bar{z}_\ell|^2 w^2(z_\ell,\bar{z}_\ell) \times \left[\bar{z}^m\prod_{\ell=1}^{n}(z-z_\ell)(z-\bar{z}_\ell) - z^m\prod_{\ell=1}^{n}(\bar{z}-z_\ell)(\bar{z}-\bar{z}_\ell)\right].$$

As soon as

$$\prod_{k<\ell}|z_k-z_\ell|^2|z_k-\bar{z}_\ell|^2\prod_{\ell=1}^{n}(\bar{z}_\ell-z_\ell)\prod_{\ell=1}^{n}(z-z_\ell)(z-\bar{z}_\ell) = \det\begin{pmatrix} 1 & z_1 & \cdots & z_1^{2n-1} & z_1^{2n} \\ 1 & \bar{z}_1 & \cdots & \bar{z}_1^{2n-1} & \bar{z}_1^{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & z_n & \cdots & z_n^{2n-1} & z_n^{2n} \\ 1 & \bar{z}_n & \cdots & \bar{z}_n^{2n-1} & \bar{z}_n^{2n} \\ 1 & z & \cdots & z^{2n-1} & z^{2n} \end{pmatrix}$$

we derive

$$\langle q_{2n}, z^m\rangle_S \propto \int\cdots\int \mathrm{d}^2Z_1\cdots\mathrm{d}^2Z_n\,\mathrm{d}^2Z_{n+1}\prod_{\ell=1}^{n+1}(z_\ell-\bar z_\ell)w^2(z_\ell,\bar z_\ell)$$

$$\times\det\begin{pmatrix}1 & z_1 & \cdots & z_1^{2n} & 0\\ 1 & \bar z_1 & \cdots & \bar z_1^{2n} & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 1 & z_n & \cdots & z_n^{2n} & 0\\ 1 & \bar z_n & \cdots & \bar z_n^{2n} & 0\\ 1 & z_{n+1} & \cdots & z_{n+1}^{2n} & z_{n+1}^m\\ 1 & \bar z_{n+1} & \cdots & \bar z_{n+1}^{2n} & \bar z_{n+1}^m\end{pmatrix}$$

where we have introduced $z_{n+1}=z$. Since a particular enumeration of z_ℓ ($1\leqslant\ell\leqslant n+1$) is irrelevant, this reduces to

$$\langle q_{2n}, z^m\rangle_S \propto \int\cdots\int \mathrm{d}^2Z_1\cdots\mathrm{d}^2Z_n\,\mathrm{d}^2Z_{n+1}\prod_{\ell=1}^{n+1}(z_\ell-\bar z_\ell)w^2(z_\ell,\bar z_\ell)$$

$$\times\det\begin{pmatrix}1 & z_1 & \cdots & z_1^{2n} & z_1^m\\ 1 & \bar z_1 & \cdots & \bar z_1^{2n} & \bar z_1^m\\ \vdots & \vdots & & \vdots & \vdots\\ 1 & z_{n+1} & \cdots & z_{n+1}^{2n} & z_{n+1}^m\\ 1 & \bar z_{n+1} & \cdots & \bar z_{n+1}^{2n} & \bar z_{n+1}^m\end{pmatrix}.$$

The latter integrand obviously vanishes for $0\leqslant m\leqslant 2n$ thus completing the proof of equation (18).
(ii) Consider

$$\langle q_{2n+1}, z^m\rangle_S \propto \int \mathrm{d}^2Z(\bar z-z)w^2(z,\bar z)\int\cdots\int \mathrm{d}^2Z_1\cdots\mathrm{d}^2Z_n\prod_{k<\ell}|z_k-z_\ell|^2|z_k-\bar z_\ell|^2$$
$$\times\prod_{\ell=1}^{n}|z_\ell-\bar z_\ell|^2w^2(z_\ell,\bar z_\ell)\left[\bar z^m\left(z+\sum_{k=1}^{n}(z_k+\bar z_k)\right)\prod_{\ell=1}^{n}(z-z_\ell)(z-\bar z_\ell)\right.$$
$$\left.-\,z^m\left(\bar z+\sum_{k=1}^{n}(z_k+\bar z_k)\right)\prod_{\ell=1}^{n}(\bar z-z_\ell)(\bar z-\bar z_\ell)\right].$$

Invoking reasoning we have used in (i), this is further reduced to

$$\langle q_{2n+1}, z^m\rangle_S \propto \int\cdots\int \mathrm{d}^2Z_1\cdots\mathrm{d}^2Z_n\,\mathrm{d}^2Z_{n+1}\prod_{\ell=1}^{n+1}(z_\ell-\bar z_\ell)w^2(z_\ell,\bar z_\ell)$$

$$\times\left[\det\begin{pmatrix}1 & z_1 & \cdots & z_1^{2n} & z_1^m\\ 1 & \bar z_1 & \cdots & \bar z_1^{2n} & \bar z_1^m\\ \vdots & \vdots & & \vdots & \vdots\\ 1 & z_{n+1} & \cdots & z_{n+1}^{2n} & z_{n+1}^m\\ 1 & \bar z_{n+1} & \cdots & \bar z_{n+1}^{2n} & \bar z_{n+1}^m\end{pmatrix}\sum_{\ell=1}^{n+1}(z_\ell+\bar z_\ell)-\det\begin{pmatrix}1 & z_1 & \cdots & z_1^{2n} & z_1^{m+1}\\ 1 & \bar z_1 & \cdots & \bar z_1^{2n} & \bar z_1^{m+1}\\ \vdots & \vdots & & \vdots & \vdots\\ 1 & z_{n+1} & \cdots & z_{n+1}^{2n} & z_{n+1}^{m+1}\\ 1 & \bar z_{n+1} & \cdots & \bar z_{n+1}^{2n} & \bar z_{n+1}^{m+1}\end{pmatrix}\right].$$

The latter trivially vanishes for $0\leqslant m\leqslant 2n-1$. This completes our proof of equation (19).

One may also verify that the normalization r_n in equations (10) and (11) is related to A_n (equation (20)) as $r_n=A_{n+1}/A_n$.

4.2. Gaussian weight

While the representations obtained above are fairly useful to study, e.g. asymptotic properties of general skew-orthogonal polynomials and address the issue of universality of eigenvalue correlations in non-Hermitean random matrix theory at $\beta = 4$, there is no need to resort to them for a simple Gaussian weight[4]

$$w_G^2(z, \bar{z}) = \exp\left[-\frac{N}{1-\tau^2}\left(z\bar{z} - \frac{\tau}{2}(z^2 + \bar{z}^2)\right)\right] \tag{21}$$

that we will be interested in following.

In this case, skew-orthogonal Hermite polynomials are simple:

$$q_{2k+1}(z) = \left(\frac{\tau}{2N}\right)^{k+1/2} H_{2k+1}\left(z\sqrt{\frac{N}{2\tau}}\right), \tag{22}$$

$$q_{2k}(z) = \left(\frac{2}{N}\right)^k k! \sum_{\ell=0}^{k} \left(\frac{\tau}{2}\right)^{\ell} \frac{1}{(2\ell)!!} H_{2\ell}\left(z\sqrt{\frac{N}{2\tau}}\right). \tag{23}$$

Here $H_k(z)$ are 'conventional' Hermite polynomials

$$H_k(z) = \frac{2^k}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}t\, \mathrm{e}^{-t^2} (z + \mathrm{i}t)^k$$

orthogonal in the complex plane z with respect to the measure $w^2(z, \bar{z})\, \mathrm{d}^2 Z$ (Di Francesco *et al* 1994):

$$\int \mathrm{d}^2 Z w_G^2(z, \bar{z}) H_k\left(z\sqrt{\frac{N}{2\tau}}\right) H_\ell\left(\bar{z}\sqrt{\frac{N}{2\tau}}\right) = \frac{\pi(1-\tau^2)^{1/2}}{N} \frac{2^k k!}{\tau^k} \delta_{k\ell}.$$

Indeed, straightforward calculation in equations (10) and (11) confirms that skew-orthogonality is met with

$$r_k = 2\pi(1-\tau)^{3/2}(1+\tau)^{1/2} \frac{(2k+1)!}{N^{2k+2}}. \tag{24}$$

Yet another, integral representation for $q_{2k}(z)$ holds which is more suitable for our purposes. To derive it we introduce the function

$$F_k(z) = \sum_{\ell=0}^{k} \left(\frac{\tau}{2}\right)^{\ell} \frac{1}{(2\ell)!!} H_{2\ell}(z) \tag{25}$$

and note that it satisfies the differential equation

$$(1+\tau)\frac{\partial F_k}{\partial z} - 2\tau z F_k(z) = -\tau \left(\frac{\tau}{2}\right)^k \frac{1}{(2k)!!} H_{2k+1}(z).$$

The latter is readily verified by using the identity $H'_\ell(z) = 2\ell H_{\ell-1}(z)$. Integrating out, we infer

$$F_k(z) = \exp\left(\frac{\tau z^2}{1+\tau}\right)\left[\sigma_k - \frac{\tau}{1+\tau}\left(\frac{\tau}{2}\right)^k \frac{1}{(2k)!!} \int_0^z \mathrm{d}z' \exp\left(-\frac{\tau z'^2}{1+\tau}\right) H_{2k+1}(z')\right]$$

[4] This weight may be thought of as originating from the matrix model $H = H_1 + \mathrm{i}v H_2$, with each of $H_\sigma (\sigma = 1, 2)$ being drawn from statistically independent Gaussian symplectic ensembles of Hermitean random matrices $P[H_\sigma] \propto \exp\{-[N/(1+\tau^2)]\,\mathrm{Tr}\,(H_\sigma^2)\}$; the parameter $v^2 = (1-\tau)/(1+\tau)$ (see, e.g., Fyodorov *et al* (1997)).

where

$$\sigma_k = \sum_{\ell=0}^{k} \left(\frac{\tau}{2}\right)^{\ell} \frac{H_{2\ell}(0)}{(2\ell)!!} \qquad H_{2\ell}(0) = (-1)^{\ell} \frac{(2\ell)!}{\ell!}.$$

Summation over ℓ can be performed explicitly resulting in

$$\sigma_k = \frac{1}{\sqrt{1+\tau}} \left[1 - \frac{\tau^{k+1}}{2^{2k+2} k!} H_{2k+2}(0) \int_0^1 \frac{\mathrm{d}\xi\, \xi^k}{\sqrt{1+\tau\xi}} \right].$$

Taken together with equations (23) and (25), this brings us to an exact integral representation for the even-order skew-orthogonal polynomials:

$$q_{2k}(z) = \left(\frac{2}{N}\right)^k \frac{k!}{\sqrt{1+\tau}} \exp\left(\frac{Nz^2}{2(1+\tau)}\right) \left\{ \left[1 - \frac{\tau^{k+1}}{2^{2k+2} k!} H_{2k+2}(0) \int_0^1 \frac{\mathrm{d}\xi\, \xi^k}{\sqrt{1+\tau\xi}} \right] - \frac{\tau}{\sqrt{1+\tau}} \left(\frac{\tau}{2}\right)^k \frac{1}{(2k)!!} \sqrt{\frac{N}{2\tau}} \int_0^z \mathrm{d}w \exp\left(-\frac{Nw^2}{2(1+\tau)}\right) H_{2k+1}\left(w\sqrt{\frac{N}{2\tau}}\right) \right\}. \tag{26}$$

5. Eigenvalue correlations in β = 4 Gaussian ensembles

In this section, we apply our findings to explicitly work out the n-point correlation function for $\beta = 4$ non-Hermitean random matrix ensemble associated with the Gaussian weight $w_G^2(z, \bar{z})$ (equation (21)). By letting the parameter τ tend to zero, a strongly non-Hermitean Ginibre's ensemble is recovered. Scaling τ with matrix size N as $\tau = 1 - \alpha^2/2N$ where $\alpha \sim O(1)$ one accesses a regime of weak non-Hermiticity (Fyodorov *et al* 1997a, 1997b) which is known to coincide with a zero-dimensional sector of a supersymmetry theory of disordered systems with a direction (Efetov 1997).

For other papers addressing non-Hermitean Gaussian ensembles of symplectic symmetry by field-theoretic (σ-model) techniques see, e.g., a supersymmetry treatment by Kolesnikov and Efetov (1999) and a replica approach by Nishigaki and Kamenev (2002). Unfotunately, both techniques run into obstacles when one attempts to study the n-point correlation function whilst replica σ-models seem to reliably provide asymptotic expansions only (Verbaarschot and Zirnbauer 1985, Kanzieper 2001).

5.1. *Finite-N solution*

In accordance with equation (6), the prekernel (equation (7)) is the only entity needed to evaluate the n-point correlation function. Equations (13), (22), (23) and (24) furnish the desired solution

$$\kappa_N(z, z') = \frac{1}{2\pi} \left(\frac{1}{1+\tau}\right)^{1/2} \left(\frac{N}{1-\tau}\right)^{3/2} \left[\sum_{k=0}^{N-1} \left(\frac{\tau}{2}\right)^{k+1/2} \frac{1}{(2k+1)!!} \times H_{2k+1}\left(z\sqrt{\frac{N}{2\tau}}\right) \sum_{\ell=0}^{k} \left(\frac{\tau}{2}\right)^{\ell} \frac{1}{(2\ell)!!} H_{2\ell}\left(z'\sqrt{\frac{N}{2\tau}}\right) - (z \leftrightarrow z') \right]. \tag{27}$$

It holds for arbitrary finite N.

5.2. *Limit of infinite matrices: $N \to \infty$*

The large-N limit is different for weakly and strongly non-Hermitean regimes.

5.2.1. Strong non-Hermiticity. As $\tau \to 0$, the prekernel simplifies to

$$\kappa_N(z, z') = \frac{N^{3/2}}{2\pi} \sum_{k=0}^{N-1} \left[\frac{(z\sqrt{N})^{2k+1}}{(2k+1)!!} \sum_{\ell=0}^{k} \frac{(z'\sqrt{N})^{2\ell}}{(2\ell)!!} - (z \leftrightarrow z') \right]. \tag{28}$$

We are interested in a thermodynamic limit $N \to \infty$ with a blown-up energy resolution $z \mapsto z/\delta_N$ where $\delta_N = (N/2\pi)^{1/2}$. To this end we have to evaluate

$$\lim_{N\to\infty} \frac{1}{\delta_N^3} \kappa_N \left(\frac{z}{\delta_N}, \frac{z'}{\delta_N} \right) = \sqrt{2\pi} \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \left[\frac{(z\sqrt{2\pi})^{2k+1}}{(2k+1)!!} \frac{(z'\sqrt{2\pi})^{2\ell}}{(2\ell)!!} - (z \leftrightarrow z') \right]. \tag{29}$$

An extra power of δ_N in the denominator of the lhs is brought about by a prefactor $(\bar{z} - z)^{1/2} (\bar{z}' - z')^{1/2}$ in equation (7).

Double summation in equation (29) can be performed explicitly. Denoting

$$\sigma(z, z') = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \left[\frac{z^{2k+1}}{(2k+1)!!} \frac{(z')^{2\ell}}{(2\ell)!!} - (z \leftrightarrow z') \right]$$

we observe that

$$\frac{\partial \sigma}{\partial z} = z\sigma + \mathrm{e}^{zz'} \qquad \frac{\partial \sigma}{\partial z'} = z'\sigma - \mathrm{e}^{zz'}.$$

This suggests that we look for $\sigma(z, z')$ in the form

$$\sigma(z, z') = \mathrm{e}^{\frac{1}{2}(z^2 + z'^2)} \Lambda(z, z').$$

As soon as

$$\frac{\partial \Lambda}{\partial z} = \mathrm{e}^{-\frac{1}{2}(z-z')^2} \qquad \frac{\partial \Lambda}{\partial z'} = -\,\mathrm{e}^{-\frac{1}{2}(z-z')^2}$$

we obtain

$$\Lambda(z, z') = \int_0^{z-z'} \mathrm{d}t\, \mathrm{e}^{-t^2/2} = \sqrt{\frac{\pi}{2}} \operatorname{erf}\left(\frac{z - z'}{\sqrt{2}} \right).$$

This results in

$$\lim_{N\to\infty} \frac{1}{\delta_N^3} \kappa_N \left(\frac{z}{\delta_N}, \frac{z'}{\delta_N} \right) = \pi \exp[\pi(z^2 + z'^2)] \operatorname{erf}[\sqrt{\pi}(z - z')]. \tag{30}$$

The latter is sufficient to evaluate all n-point correlation functions by means of equations (6) and (7). For instance, the scaled density of states (equations (5) and (14)) reads

$$\begin{aligned} \rho_1^{(4)}(z) &= (\bar{z} - z) \lim_{N\to\infty} \frac{1}{\delta_N^3} w^2 \left(\frac{z}{\delta_N}, \frac{\bar{z}}{\delta_N} \right) \kappa_N \left(\frac{z}{\delta_N}, \frac{\bar{z}}{\delta_N} \right) \\ &= 8\pi Y^2 \exp(-4\pi Y^2) \int_0^1 \mathrm{d}\lambda \exp(4\pi Y^2 \lambda^2), \end{aligned} \tag{31}$$

$Y = \operatorname{Im} z$. A particular rescaling used in equation (29) has been chosen in such a way that the scaled level density $\rho_1^{(4)}(z)$ approaches unity at infinity, $|Y| \to \infty$.

5.2.2. Weak non-Hermiticity. The regime of weak non-Hermiticity is of particular interest due to its close relation to Efetov's model of disordered systems with a direction. We reiterate that a degree of (weak) non-Hermiticity is governed by a parameter τ which scales with the matrix size N as $\tau = 1 - \alpha^2/2N$, $\alpha \sim O(1)$.

The large-N limit of the sum equation (13) (or equation (27)) is dominated by contributions of terms with k such that $k/N \sim O(1)$. One therefore needs the asymptotics of skew orthogonal polynomials $q_k(z)$ at large indices k.

Asymptotics for odd-order skew-orthogonal polynomials q_{2k+1} are those of Hermite polynomials H_{2k+1} (equation (22)). Utilizing the result from standard reference book by Szegö (1939)

$$H_{2k+1}\left(\frac{z}{2\sqrt{k}}\right) \simeq \frac{2^{2k+1}(-1)^k k!}{\sqrt{\pi k}} \sin z$$

we conclude that

$$q_{2k+1}(z) \simeq \frac{2^{2k+1}(-1)^k k!}{\pi^{1/2}} \left(\frac{\tau}{2N}\right)^{k+1/2} \sin\left(z\sqrt{\frac{2kN}{\tau}}\right). \tag{32}$$

Here $k/N \sim O(1)$ and $zN \sim O(1)$.

Asymptotics for even-order skew-orthogonal polynomials q_{2k} can be read out from equation (26). Since

$$\int_0^1 \frac{\mathrm{d}\xi\, \xi^k}{\sqrt{1+\tau\xi}} \simeq \frac{1}{k\sqrt{1+\tau}} \qquad k \gg 1$$

we derive

$$q_{2k}(z) \simeq \left(\frac{2}{N}\right)^k \frac{k!}{\sqrt{1+\tau}} \left[1 - \frac{\tau}{\sqrt{1+\tau}} \left(\frac{\tau}{2}\right)^{k+1} \frac{1}{(2k+2)!!} H_{2k+2}\left(z\sqrt{\frac{N}{2\tau}}\right)\right]. \tag{33}$$

Applying further the asymptotic formula (Szegö 1939)

$$H_{2k}\left(\frac{z}{2\sqrt{k}}\right) \simeq \frac{2^{2k}(-1)^k k!}{\sqrt{\pi k}} \cos z$$

we deduce

$$q_{2k}(z) \simeq \left(\frac{2}{N}\right)^k \frac{k!}{\sqrt{1+\tau}} \left[1 + \frac{(-1)^k \tau^{k+1}}{\sqrt{1+\tau}\sqrt{\pi k}} \cos\left(z\sqrt{\frac{2kN}{\tau}}\right)\right]. \tag{34}$$

Equations (32) and (34) for skew-orthogonal Hermite polynomials at $k \gg 1$ now make it possible to evaluate the large-N prekernel. Substituting the two equations into equation (13), and replacing the sum over k by an integral we come up with

$$\lim_{N\to\infty} \frac{1}{\delta_N^3} \kappa_N\left(\frac{z}{\delta_N}, \frac{z'}{\delta_N}\right) = -\frac{\pi^{3/2}}{4\alpha^3} \int_0^1 \frac{\mathrm{d}\lambda}{\lambda}\, \mathrm{e}^{-\alpha^2\lambda^2} \sin[\pi(z-z')\lambda]. \tag{35}$$

When taking the limit $N \to \infty$, the scale δ_N has been set to $\delta_N = N\sqrt{2}/\pi$.

In accordance with equation (6), knowledge of the scaled prekernel equation (35) is self-sufficient to have evaluated all n-point correlation functions. For instance, the density of states reads (equations (5) and (14))

$$\rho_1^{(4)}(z) = \frac{\pi^{3/2}}{2\alpha^3} Y \exp\left(-\pi^2 Y^2/\alpha^2\right) \int_0^1 \frac{\mathrm{d}\lambda}{\lambda} \exp\left(-\alpha^2\lambda^2\right) \sinh\left[2\pi Y\lambda\right]. \tag{36}$$

6. Conclusions

A problem of eigenvalue correlations in symplectic ensembles of non-Hermitean random matrices has been exactly solved by the method of orthogonal polynomials. In close analogy with $\beta = 4$ Hermitean matrix ensembles, the n-point correlation function is given by a quaternion determinant (equation (6)) of an $n \times n$ matrix whose entries are quaternions with an image given by 2×2 matrices in the form of equation (7). To evaluate the latter it is convenient (but not obligatory) to introduce a set of polynomials which are skew-orthogonal in the complex plane (equations (10) and (11)). The skew-orthogonality set by equation (12) represents a natural basis in which calculational technology is most economic.

In Gaussian random matrix ensembles, the eigenvalue correlations are described by the prekernel equation (27) which further simplifies down to equations (30) and (35) for strong and weak non-Hermiticity, respectively. These results do not apply very close to the spectrum edges, which may also be studied within the current framework.

Remarkably, at $\beta = 4$, all n-point spectral correlation functions exhibit a peculiar depletion of eigenvalues along the real axis $\operatorname{Im} z_\ell = 0$, $1 \leqslant \ell \leqslant n$, where correlations vanish for both arbitrary matrix size N and a probability measure $w^2(z, \bar{z})$. As for the remaining nontrivial functional dependence, we expect it to be universal as well once a thermodynamic limit is taken. Equations (18) and (19) will obviously serve as a proper starting point to address the universality issue in either the spectrum bulk or near the complex edges of the eigenvalue support.

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References

de Bruijn N G 1955 On some multiple integrals involving determinants *J. Ind. Math. Soc.* **19** 133
Dyson F J 1972 Quaternion determinants *Helv. Phys. Acta* **45** 289
Edelman A 1997 The probability that a random real Gaussian matrix has *k* real eigenvalues, related distributions, and the circular law *J. Mult. Analysis* **60** 203
Efetov K B 1997 Quantum disordered systems with a direction *Phys. Rev.* B **56** 9630
Eynard B 2001 Asymptotics of skew orthogonal polynomials *J. Phys. A: Math. Gen.* **34** 7591
Di Francesco F, Gaudin M, Itzykson C and Lesage F 1994 Laughlin wave-functions, Coulomb gases and expansions of the discriminant *Int. J. Mod. Phys.* A **9** 4257
Fyodorov Y V, Khoruzhenko B A and Sommers H-J 1997a Almost Hermitean random matrices: eigenvalue density in the complex plane *Phys. Lett.* A **226** 46
Fyodorov Y V, Khoruzhenko B A and Sommers H-J 1997b Almost Hermitean random matrices: crossover from Wigner–Dyson to Ginibre eigenvalue statistics *Phys. Rev. Lett.* **79** 557
Fyodorov Y V, Khoruzhenko B A and Sommers H-J 1998 Universality in the random matrix spectra in the regime of weak non-Hermiticity *Ann. Inst. H Poincaré (Physique Theorique)* **68** 449
Ginibre J 1965 Statistical ensembles of complex, quaternion, and real matrices *J. Math. Phys.* **6** 440
Halasz M A, Osborn J C and Verbaarschot J J M 1997 Random matrix triality at nonzero chemical potential *Phys. Rev.* D **56** 7059
Hastings M B 2000 Fermionic mapping for eigenvalue correlation functions of weakly non-Hermitian symplectic ensemble *Nucl. Phys.* B **572** 535
Kanzieper E and Freilikher V 1999 Spectra of large random matrices: a method of study *Diffuse Waves in Complex Media (NATO ASI, Series C (Math. and Phys. Sciences))* vol 531 ed by J-P Fouque (Dordrecht: Kluwer) p 165
Kanzieper E 2001 Random matrix theory and the replica method *Nucl. Phys.* B **596** 548
Kolesnikov A V and Efetov K B 1999 Distribution of complex eigenvalues for symplectic ensembles of non-Hermitian matrices *Waves Random Media* **9** 71
Lehmann N and Sommers H-J 1991 Eigenvalue statistics of random real matrices *Phys. Rev. Lett.* **67** 941

Mahoux G and Mehta M L 1991 A method of integration over matrix variables: IV *J. Physique I* **1** 1093
Mehta M L 1967 *Random Matrices* (New York: Academic)
Mehta M L and Gaudin M 1960 On the density of eigenvalues of a random matrix *Nucl. Phys.* B **18** 420
Nishigaki S M and Kamenev A 2002 Replica treatment of non-Hermitian disordered Hamiltonians *J. Phys. A: Math. Gen.* **35** 4571
Oas G 1997 Universal cubic eigenvalue repulsion for random normal matrices *Phys. Rev.* E **55** 205
Szegö G 1939 *Orthogonal Polynomials* (Providence, RI: AMS)
Tracy C A and Widom H 1998 Correlation functions, cluster functions and spacing distributions for random matrices *J. Stat. Phys.* **92** 809
Verbaarchot J J M and Zirnbauer M R 1985 Critique of the replica trick *J. Phys. A: Math. Gen.* **18** 1093